X-553-67-366

LOCAL ERROR CONTROL AND ITS EFFECTS ON THE OPTIMIZATION OF ORBITAL INTEGRATION

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AUGUST 1967

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INTRODUCTION

Many computing systems which are used to determine the orbits of artificial earth satellites require numerical integration. Due to advances in the theory of perturbations and the concomitant increase in the complexity of the mathematical model, highly efficient integration techniques are desirable and, in certain cases, necessary.

The orbital equations under consideration are of the form

$$\ddot{x} = \frac{-\mu x}{\|x\|^3} + P \qquad x(t_0) = x_0 , \tag{1}$$

in three space variables, where $\|x\| = (x^2 + y^2 + z^2)^{1/2}$, P is the perturbation function and μ is a constant. We will assume throughout that P is fairly complicated so that the efficiency of the integration is proportional to the total number of derivative evaluations. (Also, for simplicity we assume that P is independent of the first derivative $\dot{x}$, i.e., $P = P(t, x)$; see Appendix B.)

The numerical method under consideration is a multistep process, i.e., a method which approximates the solution of (1) by using formulas of the form

$$\sum_{i=0}^{k} \alpha_i x_{n-i} = h^2 \sum_{i=0}^{k} \beta_i \ddot{x}_{n-i} \qquad n = k, k+1, \cdots, \tag{2}$$

where h is the stepsize, $x_i = x(t_0 + ih)$, and α_i, β_i are constants. Formulas of this type define a linear k-step method (see Henrici 1962, pp. 295 ff.), and are usually derived by the integration of a polynomial approximation of the derivative $\ddot{x}$.

Associated with (2) is a local truncation error of the form

$$R_n = R(t_n) = C h^{p+2} x^{(p+2)}(\xi) , \qquad (3)$$

where $\xi \in [t_{n-k}, t_n]$, C is a constant, and p is an integer called the order of the method, all depending on k, the number of "backpoints" used.

It has been suggested (Soar, 1964) that controlling the local error by varying the order and stepsize during the integration of the equations of motion (1) may yield gains in efficiency without sacrificing accuracy. For example, it is clear that the magnitude of (3) is sensitive to variations in p or h. Suppose that during the integration, the magnitude of R_n becomes insignificant relative to the calculations being performed. Then an increase in the magnitude of h or a decrease in the magnitude of p, either separately or in combination, may increase the efficiency of the integration without sacrificing accuracy.

The purpose of this study then, is to develop techniques to automatically control the magnitude of R_n during the integration by varying the parameters p and h, and to examine the effects* of these controls on the efficiency and accuracy of the process.

Before discussing methods of estimating and controlling the local error, we formulate a commonly used integration model which was used to obtain the results.

THE INTEGRATION FORMULAE

We begin by remarking that in equation (2), if $\beta_0 \neq 0$, then knowledge of the solution x_n is required on both sides of the equation and is, in general, not explicitly solvable. Equations of this type (closed form) however, have smaller associated truncation errors as well as desirable stabilizing characteristics; (Hull and Creemer, 1963). The well-known predictor-corrector algorithm utilizes formulas of this type by first computing an initial (predicted) approximation of the solution using a similar formula with $\beta_0 = 0$, then using the closed form iteratively until convergence is achieved. It can be shown that for sufficiently small h, the successive "corrected" values obtained by this process

*It should be noted that local error control does not in general yield a quantitative appraisal of the accumulated error. A qualitative control of the accumulated error is possible though, and in most cases is sufficient. For a discussion on accumulated error estimations, see Henrici, 1962.

converge to the unique solution of the closed form equation provided the function being integrated is sufficiently smooth.

Consider now the predictor-corrector formulas which are derivable from Newton's backward difference interpolation polynomial (Henrici 1962, pp. 291-293):

$$\text{(Pred)}\quad x_n + 2x_{n-1} + x_{n-2} = \nabla^2 x_n = h^2 \left[1 + \frac{1}{12}\nabla^2 + \frac{1}{12}\nabla^3 + \frac{19}{240}\nabla^4 + \frac{3}{40}\nabla^5 + \cdots + \sigma_q \nabla^q\right] \ddot{x}_{n-1} \tag{4}$$

$$\text{(Corr)}\quad x_n - 2x_{n-1} + x_{n-2} = \nabla^2 x_n = h^2 \left[1 - \nabla + \frac{1}{12}\nabla^2 - \frac{1}{240}\nabla^4 - \frac{1}{240}\nabla^5 + \cdots + \sigma_q^* \nabla^q\right] \ddot{x}_n \tag{5}$$

These are the Störmer-Cowell formulae expressed in terms of backward differences. The local truncation errors associated with these formulae are given by (3) where $p = q + 1$ and $C = \sigma_{q+1}$ or σ^*_{q+1}. We note that if the parameter q is fixed, (and hence the order p), then (4) and (5) can be written in the form of (2) by expressing the backward differences in terms of the ordinates $\ddot{x}_i$.

In particular, for $q = 5$ we have the Störmer-Cowell formulas of order 6:

$$\nabla^2 x_n = \frac{h^2}{240}\left[317\,\ddot{x}_{n-1} - 266\,\ddot{x}_{n-2} + 374\,\ddot{x}_{n-3} - 276\,\ddot{x}_{n-4} + 109\,\ddot{x}_{n-5} - 3\,\ddot{x}_{n-6}\right] \tag{4}'$$

$$\nabla^2 x_n = \frac{h^2}{240}\left[18\,\ddot{x}_n + 209\,\ddot{x}_{n-1} + 4\,\ddot{x}_{n-2} + 14\,\ddot{x}_{n-3} - 6\,\ddot{x}_{n-4} + \ddot{x}_{n-5}\right] \tag{5}'$$

It is clear that the coefficients in (4)' and (5)' depend on the choice of order, so that varying the order during the integration would mean producing a new set of coefficients for each p. It is for this reason that formulas of the type (4) and

(5), where variations of the order can be made simply by varying the number of terms retained, were used in this study.(*)

We note that before any of the above formulas could be used, a set of "starting" values of the solution must be computed.(**) For example, for equations (4)' and (5)', if the values x_i (and hence $\ddot{x}_i$), $i = 0, 1, \cdots 5$ are known, then (4)' could be used to obtain a "predicted" value x_6^p $(n = 6)$ and (5)' to obtain the successive corrections $x_6^{c_j}$, $j = 1, 2, 3, \cdots$ until convergence to same criterion δ is achieved, i.e., until

$$|x_6^{c_{j+1}} - x_6^{c_j}| \leq \delta \ .$$

The process could then be repeated with $n = 7, 8, \cdots\cdots$.

LOCAL ERROR ESTIMATION

Any considerations concerning the control of the local error depend on the capability of obtaining a reasonable estimate of (3) during the integration. One of the most widely used approximations is based on the fact that if $f(x)$ is an n-times continuously differentiable function, then there exists a θ, $0 < \theta < 1$, such that

$$\frac{\Delta^n f(x)}{(\Delta x)^n} = \frac{d^n f}{dx^n}(x + n\theta\,(\Delta x)), \quad \text{i.e.,}$$

if Δx is sufficiently small, we can approximate a high order derivative by a high order difference. In particular, we can write

$$R_n = R(t_n) = Ch^{p+2}\, x^{(p+2)}(\xi) = Ch^{p+2}\, \ddot{x}^{(p)}(\xi) \cong Ch^2 \nabla^p \ddot{x}_n \qquad (6)$$

(*) In actual practice, a modification of equations (4) and (5) known as the "summed" form of the integration formulae was used. This modification is formulated in Appendix A.

(**) Experimentation has indicated that recently developed high order Runge-Kutta type formulas are particularly suited for such purposes; see Shanks, 1966.

If suitable predictor-corrector formulas are used, another technique available is based on comparing the predicted value x_n^p with the finally accepted "corrected" value x_n^c; in particular, it can be proven (Henrici, 1962) that for equations (4) and (5),

$$R_n \cong K\left(x_n^c - x_n^p\right), \tag{7}$$

where K is a constant. This technique is known as Milne's method of estimating the local error.

LOCAL ERROR CONTROL

We consider now a specific technique which would enable one to control the magnitude of the local error during the integration by varying the parameters p and h. Let T_1 and T_2 be tolerances specifying the desired upper and lower bounds on the local error, so that for any n, the local error must satisfy

$$T_2 \leq |R_n| \leq T_1, \quad \text{where} \quad T_2 \leq T_1 . \tag{8}$$

Controlling the local error by varying the order can then be accomplished by determining whether condition (8) is satisfied for each n, i.e., if for some n, $R_n < T_2$ decrease the order, or if $R_n > T_1$, increase the order.

We note that varying the order alone is generally not sufficient for an effective control. For example, if for some n, $R_n > T_1$ and h is large, it may not be possible to satisfy (8) with any p, in which case the stepsize must be decreased. (Also, the danger of numerical instability increases considerably with large p; see references 5 and 6.) On the other hand, $R_n < T_2$ and p small would indicate that a larger stepsize could be used.

Controlling the local error by varying the stepsize, in addition to varying the order, can be accomplished as follows: Let L_1 and L_2 be limits specifying the desired upper and lower bounds on the order, i.e.,

$$L_2 \leq p \leq L_1 \quad \text{where} \quad L_2 \leq L_1 ; \tag{9}$$

then, if at some point during the integration, $p > L_1$, decrease the stepsize, or if $p < L_2$, increase the stepsize.

A local error control designed in this way has the property that the parameters T_1, T_2, L_1, L_2 completely govern the degree and type of control; i.e., the parameters T_1 and T_2 can be selected so as to effect any degree of control from none, to a continuous step by step control; likewise, the parameters L_1 and L_2 can be selected so that the control involves varying the order alone, varying the step alone, or varying both. For example, if $L_1 = L_2$, the control depends solely on variations in the stepsize.

We remark that unlike varying the order, changes in the magnitude of the stepsize during the integration are usually not easily accomplished since a "memory" of equally spaced points is required at every step during the integration. For this reason a common technique is the "halving-doubling" method where an "increase" or "decrease" in the stepsize is either half, or double the current stepsize. Then increasing the stepsize presents no problem if a sufficient number of backpoints are retained. Decreasing is usually accomplished by some interpolation technique or by using a single step method.

OPTIMIZATION OF STEPSIZE

Restricting the variations of stepsize, h, by a constant factor does not, in general, yield optimal stepsizes*, and hence the initial choice of interval could have a substantial effect on the total number of integration steps. For example, suppose the halving-doubling method is being used and the order is held fixed. If for some n, $R_n > T_1$, the stepsize must be decreased. Let h_{opt} be the optimum (largest) value of h for which (8) is satisfied. We could then have the situation $h/2 < h_{opt} < h$. Because of our restriction $h/2$ would be used, although this would result in more integration steps than would be required if the optimum stepsize were used. Hence a variation of h which would better approximate its optimum value is desirable. One technique available (see Reference 4 for details and applications in the case of single-step methods) is the computation of the stepsize using the local error estimate, i.e., let σ be the "allowable" local error for each step, where $T_2 \leq \sigma \leq T_1$. Variations in h can then be computed by using the relationship between σ and $R_n \cong Ch^{p+2}x^{(p+2)}(\xi)$, viz.,

$$h_{opt} \cong \left[\frac{\sigma}{Cx^{(p+2)}(\xi)}\right]^{1/p+2} \cong \left[\frac{\sigma h^{p+2}}{R_n}\right]^{1/p+2}, \qquad (10)$$

*The largest stepsize which allows a prescribed local error at a given point.

so that if $\sigma < R_n$, the stepsize is decreased, or if $R_n < \sigma$ the stepsize is increased, where the variation is approximately optimal with respect to the choice of σ.

OPTIMIZATION OF ORDER

As in the case of varying the stepsize, we see that varying the order by constant factor need not yield the optimal* order. For example, suppose a variable order, constant step control is being used and the order is varied by ± 1. If for some n, $|R_n| < T_2$, the order must be decreased. Let p_{opt} be the optimum (smallest) value of p for which (8) is satisfied. We could then have the situation

$$p_{opt} < p-1 < p.$$

Because of our restriction $p - 1$ would be used, although this would result in more calculations than would be required at the optimum order. Also, if we were using a variable step control in addition to varying the order, it may occur that

$$p_{opt} < L_2 \leq p - 1 ,$$

in which case the stepsize would be increased if p_{opt} were used. One method which could be used to obtain the optimum order is to test the local error for various orders until the optimum is established. For example, if the local error estimate used is given by (6), then one could test $Ch^2\nabla^p \ddot{x}_n$ for various p until the smallest value satisfying (8) is found.

GENERAL EFFICIENCY CONSIDERATIONS

Let us examine some possible effects, advantages and disadvantages of the above mentioned controls on the efficiency of any integration, which will generally depend on the following:

(a) Minimizing the number of integration steps,
(b) Minimizing the number of corrector iterations required at each step

*The smallest order which allows a prescribed local error.

(c) Minimizing other computational efforts required by the integration formulae being used, such as retaining only the significant terms in (4) or (5) during the integration.

Further, we must consider

(d) The computational effort and time involved in controlling the local error, such as producing the required "memory" when changing the stepsize.

We now examine the following controls:

(I) A variable order, fixed step control

Since varying the order involves only varying the number of terms retained, this control has little effect on (d). Equation (7) however, expressing the relationship between the local error and the "predicted-corrected" difference, indicates that controlling the local error may substantially effect the total number of corrector iterations. In particular, an increase in order at some point during the integration where the local error is increasing could minimize any increase in the number of required iterations.

Another possible advantage of this control is its obvious effect on (c), and that given a stepsize, the order required to satisfy a given criterion on the local error is automatically selected.

A clear disadvantage of this control is that it has no effect on minimizing the number of integration steps.

(II) A fixed order, variable step control

This control affects the number of corrector iterations for the same reason as (I). It is clear that (a) and (d) are affected, and we must consider the "trade-off" between these two, i.e., we must consider the gain in efficiency due to the larger stepsizes against the cost in changing the stepsize. Furthermore, we have discussed two possible methods for varying the stepsize, viz., (i) halving-doubling, and (ii) step computation using the local error estimate. In both cases

the necessary "memory" could be obtained by interpolation, but in method (ii) all the backpoints must be computed both when decreasing and increasing, as opposed half, or none, for method (i). On the other hand, we have seen that method (ii) could be more effective in minimizing the integration step than method (i).

Besides its possible effects on (a), another advantage of this control is that given an order, the stepsize (possibly optimum) required to satisfy a given criterion on the local error is automatically selected.

(III) A variable order, variable step control

The control combines the effects on (I) and (II). A possible disadvantage of this control is that in allowing the order to vary before changing the step, smaller stepsizes may result, thereby adversely affecting (a), (as opposed to control (II)).

A clear advantage of this control is that both the stepsize and order required to satisfy a given local error criterion are automatically selected.

EFFECTIVE ERROR CONTROL IN ORBITAL INTEGRATION

In general, the effectiveness of a local error control during the integration of a particular orbit depends on the degree and rate of change of the derivative $\ddot{x}^{(p+2)}(\xi)$, (and hence the local error), during a revolution for a fixed p and h. This change is governed by the orbital parameters a, the semi-major axis, and e, the eccentricity (Soar, 1964). In particular, for orbits with eccentricity near zero, it can be expected that the local error variation will be small over a revolution, and that as the eccentricity becomes bounded away from zero, this variation becomes more pronounced. Moreover, the larger the semi-major axis is for a particular satellite, the slower the rate of change of the derivative.

Assume now that during the computation of a particular orbit, some local error criterion is to be satisfied over the entire range of integration, (which may involve many revolutions), i.e., we wish to bound the local error from above, so

as to restrict, at least qualitatively, the propagation of truncation error. Assume also that the integration method used is of some fixed order, (as is generally the case when formulas of the type (4)' or (5)' are used), and that the stepsize is to be specified. In such a situation, one could insure the error criterion will be satisfied and yet obtain some degree of efficiency, by integrating over a single revolution with various stepsizes, and selecting the largest stepsize which satisfies the criterion over the entire revolution. For orbits with $e \tilde{=} 0$, this process could yield the optimal stepsize for the given order. On the other hand, for orbits with $e \geq \epsilon > 0$, the stepsize selected in this manner would correspond to bounding the local error at its maximum value over the revolution (e.g., at perigee), and could result in needless computation for those portions of the revolution where the local error is smaller, or at its minimum, (e.g., near apogee).

Consider now the effects of a continuous variable step control during the integration. In both cases cited above, i.e., $e \tilde{=} 0$, or $e \geq \epsilon > 0$, an initial stepsize (possible optimum) satisfying the local error criterion would be computed automatically, and furthermore, the stepsize would vary according to variations in the derivative. It will be seen in the numerical results that in both cases, under suitable conditions, such a control has a considerable effect on the efficiency. In particular, it will be seen that if a sufficiently large range (T_1, T_2) in (8) is used, so that the local error is allowed to range between these two limits before any interval modification is performed, the gains in efficiency due to the larger stepsizes will overshadow any loss due to changing the stepsize.

If, in addition to the above, the order to be used could be specified, (as is generally the case when formulas of the type (4) or (5) are used), then it may be possible to obtain an optimal stepsize (or stepsizes) - order combination by integrating over a single revolution with various orders and letting the stepsize vary during the revolution. In particular, one could find that order which, together with a continuous step modification, effects the most efficient integration. Examples of such a procedure will be given in the numerical results.

Finally, let us examine the effects of a continuous variable order control. We note again that since we are assuming that the number of evaluations of the derivative governs the overall efficiency, the effects on this control or (c) (in the previous section), and its possible adverse effects on (a) when used in conjunction with a variable step control, make this control ineffective from this point of view. If, however, a fixed step method is used, it will be seen that the effect of this control on the number of corrector iterations can result in considerable gains in efficiency.

NUMERICAL RESULTS

In this section the results obtained by applying the various local error controls during the integration of three selected orbit types will be examined. The formulation of the actual equations of motion (1) used is presented in Appendix B, and a description of the computer program can be found in Reference 9.

Before proceeding, we make the following remarks concerning the numerical computations:

(i) In actual practice, for simplicity of computation, instead of using an approximation for Rn (given by (6) in controlling the error, a bound on the local error given by

$$Un = C h^2 \nabla^{k-2} \ddot{x}_n$$

was used, where ∇^{k-2} is the last backward difference retained in the computations; (see Appendix A). Since $p = k + 1$, we have

$$|Rn| \leq |Un| \quad ,$$

and thus the qualitative results obtained by using Un are the same as if Rn were used, and the same quantitative results could be obtained by using Rn and a smaller T_1.

(ii) The number of derivative evaluations for each integration was obtained by multiplying the total number of steps taken by the average number of predictor-corrector iterations.

(iii) The error estimates tabulated were obtained by integrating (1) with $P = 0$, and comparing the results with the Keplerian solution. Since the effect of the perturbation function P on the accumulation of error is generally small, the error estimate obtained in this manner can be assumed to be a good estimate of the actual error generated.

(iv) All computations were performed on the Univac 1108 computer using double-precision arithmetic. In the computations the following units were used:

unit of length = 6378.388 Km.
unit of time = 13.4472 Mins.

We begin by tabulating (Table I) the results obtained by integrating the orbits with formulas of orders 7, 9, 11 and 13. These integrations were carried out with various stepsizes, where in each case the stepsize used was increased until Un (see remark (i)) failed to satisfy the inequality

$$|Un| \leq T_1 \tag{12}$$

over the entire range of integration, which was chosen arbitrarily to be 4000 mins. We have tabulated only the last 2 stepsizes tested, indicating the largest stepsize which passed the criterion, and the first stepsize which failed.

Table I

Fixed Order - Fixed Step

$T_1 = .5 \times 10^{-8}$ $\delta = .1 \times 10^{-10}$ (predictor-corrector tolerance)

a	e	Order	Stepsize (Mins.)	No. of Der. Eval.	Estimated Error
6.7	.003	7	5.0	798	$.4 \times 10^{-9}$
			7.0*	569	$.4 \times 10^{-8}$
		9	15.0	262	$.3 \times 10^{-8}$
			17.0*	231	$.8 \times 10^{-8}$
		11	24.0	160	$.9 \times 10^{-9}$
			26.0*	147	$.2 \times 10^{-8}$
		13	22.0	173	$.3 \times 10^{-11}$
			24.0*	158	$.9 \times 10^{-10}$
1.15	.075	7	.4	9998	$.8 \times 10^{-7}$
			.5*	7998	$.4 \times 10^{-6}$
		9	.9	4440	$.2 \times 10^{-6}$
			1.0*	3996	$.5 \times 10^{-6}$
		11	1.2	3327	$.5 \times 10^{-7}$
			1.3*	3070	$.1 \times 10^{-6}$
		13	1.5	3081	$.1 \times 10^{-8}$
			1.6*	3043	$.1 \times 10^{-7}$
8.5	.878	7	.30	13344	$.1 \times 10^{-6}$
			.35*	11483	$.4 \times 10^{-6}$
		9	.40	10005	$.4 \times 10^{-7}$
			.45*	8902	$.1 \times 10^{-6}$
		11	.30	13340	$.9 \times 10^{-10}$
			.35*	11433	$.4 \times 10^{-9}$
		13	.20	19992	$.4 \times 10^{-11}$
			.25*	15992	$.3 \times 10^{-11}$

*Integration with this stepsize failed the criterion (12) for some n.

Concerning the results in Table I, we make the following observations:

(i) For a given local error criterion, increasing the order does not generally imply a larger "allowable" stepsize. Since the propagation of error (both truncation and round-off) influences the "smoothness" of the higher order differences (and hence our local error estimate), one would expect this behavior both from the inaccuracies in the differences as well as an unstable p and h combination. We also note an increase in accuracy with the higher orders. A table demonstrating this behavior (for the case $e \cong 0$) can be found in Reference 6.

(ii) The stepsizes resulting from our local error criterion during the integrations correspond to bounding the local error at its maximum (perigee), although the local error for most of the integration, particularly for the case $e = .87$, was considerably smaller than our upper bound.

In Table II the results obtained by integrating our test orbits with a variable step control are tabulated. In particular, the stepsize was varied during each integration forcing Un to satisfy

$$T_1 \leq |Un| \leq T_2 \tag{13}$$

for all n. The stepsize modification techniques used were the halving-doubling and optimum step computation where the "allowable" local error used is designated by σ (see table below); i.e., the step computation is given by

$$h_{opt} = \left[\sigma h^{p+2}/Un\right]^{1/p+2} .$$

Except for those cases indicated by an asterisk, all integrations were performed with an initial stepsize (chosen arbitrarily) of .42 mins. = $1/2^5$ in internal units. We have indicated the stepsize or stepsize range which occurred during the multistep integration as a result of our local error control.

Comparing the corresponding results in Tables I and II, we make the following observations:

(i) For the "low e" cases, ($e = .003, .075$), the stepsizes selected as a result of the variable step control are comparable to the "optimum" stepsizes found by trial and error in Table I.

(ii) The dependence on the initial choice of stepsize in the halving-doubling type of step modification is demonstrated by the cases in which the initial step was selected as a result of foreknowledge of the "optimum" step, (indicated by asterisks in Table II). In these cases, the initial step used was exactly half the stepsize indicated.

Table II

Fixed Order - Variable Step

$T_1 = .5 \times 10^{-8}$ $T_2 = .5 \times 10^{-13}$ $\delta = .1 \times 10^{-10}$

a	e	P	Type of Step Mod.	σ	Stepsize (Range)	No. of Der. Eval.	Est. Error
6.7	.003	7	H/D		1.7	2377	$.2 \times 10^{-12}$
			OPT	$.1 \times 10^{-9}$	5.4	744	$.6 \times 10^{-9}$
			OPT	$.1 \times 10^{-10}$	3.0	1320	$.1 \times 10^{-10}$
		9	H/D		6.7	590	$.2 \times 10^{-11}$
			H/D*		14.0	281	$.1 \times 10^{-8}$
			OPT	$.1 \times 10^{-9}$	12.4	322	$.5 \times 10^{-9}$
			OPT	$.1 \times 10^{-10}$	8.4	471	$.1 \times 10^{-10}$
		11	H/D		13.4	291	$.2 \times 10^{-11}$
			H/D*		23.0	167	$.6 \times 10^{-9}$
			OPT	$.1 \times 10^{-9}$	20.1	196	$.1 \times 10^{-9}$
			OPT	$.1 \times 10^{-10}$	15.1	260	$.7 \times 10^{-11}$
		13	H/D		13.4 - 26.9	174	$.6 \times 10^{-8}$
			H/D*		23.0	165	$.5 \times 10^{-11}$
			OPT	$.1 \times 10^{-9}$	19.8 - 30.0	155	$.8 \times 10^{-8}$
			OPT	$.1 \times 10^{-10}$	13.9 - 26.8	159	$.7 \times 10^{-8}$
1.15	.075	7	H/D		.42	9516	$.1 \times 10^{-6}$
			OPT	$.1 \times 10^{-9}$	.42	9516	$.1 \times 10^{-6}$
			OPT	$.1 \times 10^{-10}$	.42	9516	$.1 \times 10^{-6}$
		9	H/D		.42	9514	$.2 \times 10^{-9}$
			OPT	$.1 \times 10^{-9}$	.61	6514	$.6 \times 10^{-8}$
			OPT	$.1 \times 10^{-10}$	.42	9490	$.2 \times 10^{-9}$
		11	H/D		.84	4781	$.1 \times 10^{-8}$
			OPT	$.1 \times 10^{-9}$	1.04	3849	$.8 \times 10^{-8}$
			OPT	$.1 \times 10^{-10}$	.78	5131	$.4 \times 10^{-9}$

H/D = "Halving-doubling"
OPT = "Optimum Step Computation"

Table II (Continued)

a	e	P	Type of Step Mod.	σ	Stepsize (Range)	No. of Der. Eval.	Est. Error
		13	H/D		.84 - 1.68	4257	$.4 \times 10^{-6}$
			OPT	$.1 \times 10^{-9}$	1.20	3314	$.8 \times 10^{-9}$
			OPT	$.1 \times 10^{-10}$	.96	4171	$.5 \times 10^{-9}$
8.5	.87	7	H/D		.2 - 3.4	3180	$.7 \times 10^{-8}$
			OPT	$.1 \times 10^{-9}$	.2 - 9.5	2415	$.2 \times 10^{-7}$
			OPT	$.1 \times 10^{-10}$	.1 - 6.9	3131	$.3 \times 10^{-8}$
		9	H/D		.2 - 6.7	1374	$.7 \times 10^{-7}$
			OPT	$.1 \times 10^{-9}$	.2 - 19.0	1137	$.7 \times 10^{-7}$
			OPT	$.1 \times 10^{-10}$	.2 - 14.6	1331	$.1 \times 10^{-8}$
		11	H/D		.2 - 13.4	875	$.6 \times 10^{-7}$
			OPT	$.1 \times 10^{-9}$	.2 - 29.1	788	$.2 \times 10^{-7}$
			OPT	$.1 \times 10^{-10}$	.2 - 14.0	907	$.5 \times 10^{-8}$
		13	H/D		.2 - 26.9	710	$.1 \times 10^{-6}$
			OPT	$.1 \times 10^{-9}$	.2 - 22.4	661	$.3 \times 10^{-7}$
			OPT	$.1 \times 10^{-10}$	.1 - 25.6	775	$.1 \times 10^{-7}$

(iii) For the "high e" case, (e = .87), significant gains in efficiency due to the larger stepsizes was effected. (No loss in efficiency due to stepsize modification was noted. See the results concerning the degree of control below - Table II.) The apparent loss in accuracy, (especially for the higher orders), was expected since the error control bounded the error from below as well as above, whereas for the corresponding results in Table I, the local error was insignificant for most of the integration.

(iv) In all cases, the optimum step computation type interval modification yielded fewer derivative evaluations than halving-doubling, with little or no loss of accuracy. The effect of the magnitude of the parameter on the integration was as expected, viz., the small value yielded an increase in accuracy at a cost of smaller stepsizes.

(v) In all cases, the advantage of automatic selection of a stepsize required to satisfy the local error criterion for the given order is evident. Also note that the "best" order, (yielding the fewest derivative evaluations) to use with the variable step control may not be the same as the "best" order to use for a fixed order, fixed step integration.

We now consider the effects of a variable-order, variable step control on our test orbits. The results of these integrations are given in Table III. By the definition of this control, the order was allowed to vary between the limits L_1 and L_2 before any step modification was effected. In all cases, the "smallest" order satisfying (12) in any given stepsize was used. All integrations we performed with an initial stepsize of .42 mins. and an initial order of $L_1 + (L_2 - L_1)/2$, (forcing the initial order p to be in the interval (L_1, L_2)). We have indicated the stepsize and order range which occurred during the integration as a result of the error control.

Table III

Variable Order - Variable Step

$T_1 = .5 \times 10^{-8}$ $T_2 = .5 \times 10^{-13}$ $\delta = .1 \times 10^{-10}$ $\sigma = .1 \times 10^{-9}$

a	e	$L_1 - L_2$	Type of Step Mod.	Stepsize (Range)	Order (Range)	No. of Der. Eval.	Est. Error
6.7	.003	7 - 13	H/D	3.3	8	1184	$.9 \times 10^{-13}$
			OPT	8.8	10	454	$.5 \times 10^{-12}$
		9 - 13	H/D	6.7	9	588	$.2 \times 10^{-11}$
			OPT	18.4	11	217	$.5 \times 10^{-10}$
		11 - 15	H/D	13.4	11	289	$.2 \times 10^{-11}$
			OPT	24.6	11 - 15	175	$.2 \times 10^{-8}$
1.15	.075	7 - 13	H/D	.42	8 - 10	9513	$.9 \times 10^{-9}$
			OPT	.84	10	4767	$.5 \times 10^{-8}$
		9 - 13	H/D	.42 - .84	9 - 10	4818	$.6 \times 10^{-8}$
			OPT	1.04	11	3849	$.8 \times 10^{-8}$
		11 - 15	H/D	.84	12 - 13	4751	$.1 \times 10^{-9}$
			OPT	1.2	13	3314	$.8 \times 10^{-9}$
8.5	.87	7 - 13	H/D	.2 - 3.3	7 - 13	2744	$.4 \times 10^{-8}$
			OPT	.2 - 10.8	7 - 13	2269	$.1 \times 10^{-7}$
		9 - 13	H/D	.2 - 13.4	9 - 13	1002	$.6 \times 10^{-7}$
			OPT	.2 - 13.2	9 - 13	1067	$.3 \times 10^{-7}$
		11 - 15	H/D	.1 - 26.8	11 - 15	665	$.3 \times 10^{-6}$
			OPT	.1 - 20.0	11 - 15	742	$.2 \times 10^{-6}$

Examining these results and comparing with the previous tables we note the following:

(i) For the "low e" cases, the stepsizes attained for the various orders selected were comparable to the stepsizes obtained for the corresponding orders in Table II.

(ii) In all cases, (especially e = .87), allowing the order to vary before changing the stepsize resulted in a smaller "mean" stepsize and thus a larger number of derivative evaluations; so that if the "best" order to use with a variable step control is known, this control results in a more efficient integration than the variable order-variable step control.

(iii) In all cases, the advantage of automatic selection of both stepsize and order required to satisfy the given local error criterion is evident.

In Table IV below, the effects of a variable order control alone or the average number of predictor-corrector iterations, (and hence the total number of derivative evaluations), and the total error is demonstrated. The order or order range obtained as a result of the control is indicated.

Table IV

Variable Order-Fixed Step and Fixed Order-Fixed Step

$T_1 = .5 \times 10^{-8}$ $T_2 = .5 \times 10^{-13}$ $\delta = .1 \times 10^{-10}$

a	e	Mode	Stepsize	Order	No. of Steps	Avg. no. of p-c Iterations	Est. Error
6.7	.003	Vary Order	25.0	11	154	1.00	$.1 \times 10^{-8}$
		Fixed Order	25.0	7	158	1.80	$.4 \times 10^{-5}$
		Fixed Order	25.0	9	156	1.00	$.2 \times 10^{-6}$
1.15	.075	Vary Order	1.2	11	3327	1.00	$.5 \times 10^{-7}$
		Fixed Order	1.2	7	3331	1.94	$.1 \times 10^{-4}$
		Fixed Order	1.2	9	3329	1.28	$.2 \times 10^{-5}$
8.5	.87	Vary Order	0.5	5 - 10	8000	1.00	$.2 \times 10^{-7}$
		Fixed Order	0.5	7	7998	1.01	$.5 \times 10^{-5}$

Examining these results, we note the advantages of automatic selection of order both from the predictor-corrector point of view and from the accumulated error which results by controlling the local error.

Finally, we wish to consider the effects of varying the "allowable range" (T_1, T_2) in the local error control. In general, the smaller the interval (T_1, T_2) is made, the greater the frequency of interval and/or order modification, and one would expect, particularly in the case of a variable step control, that efficiency gains due to the larger stepsizes resulting from the control could be offset by the cost in changing the stepsize, should such changes occur too frequently during the integration.

In Table V, the results of integrating one of our test orbits with a fixed T_1 and various values for T_2 (approaching T_1 as a limiting value), are tabulated. A variable step control (halving-doubling) was used, and the variations in the stepsize, along with the number of step changes and computation time (in min.)

Table V

Variable Step Control - Range Modification

$T_1 = .5 \times 10^{-8}$ $\quad \delta = .1 \times 10^{-10}$

a	e	T_2	Stepsize (Range)	No. of Der. Eval.	Est. Error	Computation Time (Mins.)
8.5	.87	$.5 \times 10^{-15}$	.42 - 6.7	1810	$.7 \times 10^{-7}$	.094
		$.5 \times 10^{-13}$	.42 - 6.7	1309	$.7 \times 10^{-7}$	.073
		$.5 \times 10^{-11}$	.42 - 13.4	886	$.6 \times 10^{-7}$	.059
		$.1 \times 10^{-10}$	.42 - 13.4	849	$.9 \times 10^{-7}$	.058
		$.2 \times 10^{-10}$	.42 - 13.4	832	$.1 \times 10^{-6}$	.260
		$.25 \times 10^{-10}$	.42 - 26.9	856	$.1 \times 10^{-6}$	.810
		$.3 \times 10^{-10}$*	.42 - 26.9	892	$.1 \times 10^{-6}$	2.10*

are presented for each integration. We have indicated by an asterisk that integration in which a stepsize modification occurred approximately at each step, rendering the error control completely ineffective, since the entire computation time was governed by the backpoint computation.

Examining these results, we see that as T_2 approaches T_1, gains in efficiency due to the larger stepsizes is first obtained, but as the range (T_1, T_2) becomes smaller, these gains are offset by the cost in changing the step, and finally, as

expected, causes the control to be completely ineffective. We remark, however, that the equations of motion used to obtain our results (see Appendix B) were not as complicated or lengthy as one might encounter in actual practice and in a more realistic situation, a much smaller interval (T_1, T_2) may be possible before the computing time is completely governed by the frequency of step modification.

As a final note, we remark that although we have considered in our numerical results only a particular model of the perturbation function P it is expected that variations in this function, such as the inclusion of higher order gravitational effects, should not affect the general qualitative behavior of the local error and hence our results.

SUMMARY AND CONCLUSIONS

The problem of effecting an efficient and accurate orbital integration by a multistep process using the concept of controlling the local error during the integration has been studied. It has been shown that during the integration, the parameters p and h can be used to control the local error in such a way that the efficiency of the process is improved with no effective sacrifice in accuracy.

In particular, it has been shown that if a sufficiently large range (T_1, T_2) in the local error is allowed, and an order p is given, a variable step control can yield a good approximation of the optimal initial stepsize (with respect to the given order) automatically, and can significantly improve the efficiency of the process by varying this step during the integration. Moreover, if a good approximation of the "best" order to use with a variable step - fixed order control is known, this control effects the most efficient integration with respect to the various controls considered. On the other hand, a variable-order variable step control has the advantage that both the order and step required to satisfy the given local error criterion are automatically determined, and also effects a reasonably efficient integration. Finally, it has been seen that even a variable-order fixed-step control, although ineffective in minimizing the number of integration steps, can effect a more efficient integration than no control at all, by minimizing the number of predictor-corrector iterations.

We conclude by remarking that we have considered only a "local" optimization problem in the sense that the optimal stepsize is defined on a step-by-step basis as being the largest stepsize satisfying a given local error criterion at any given point. A more significant problem would be the consideration of the optimal stepsize (and/or order) distribution over the entire range of integration, defined on the basis of a criterion on the accumulated truncation error. Under some restrictive conditions, (among others, that we have only a single

differential equation), the problem of optimizing the mesh distribution is considered in a paper by D. Morrison.* One of the basic problems one would encounter in applying such techniques in our context is the requirement that a "memory" of equally spaced points be available at each step during the integration. One solution that seems worth considering is the integration of divided-difference interpolation polynomials, (which do not require equally spaced points), for use in numerical integration.

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APPENDIX A

DERIVATION OF THE "SUMMED" FORM OF INTEGRATION FORMULAE

As remarked earlier, by integrating Newton's backward difference interpolation polynomial, the following multi-step formulas for the numerical integration of (1) can be derived*, (we include formulas for the velocity in the event that P contains $\dot{x}$):

(A) Equations for $\ddot{x} \rightarrow x$

Predictor: (A1) $$x_{m+1} - 2x_m + x_{m-1} = h^2 \left[\sigma_0 \ddot{x}_m + \sigma_1 \nabla \ddot{x}_m + \sigma_2 \nabla^2 \ddot{x}_m + \sigma_3 \nabla^3 \ddot{x}_m + \cdots + \sigma_k \nabla^k \ddot{x}_m\right]$$

Corrector: (A2) $$x_{m+1} - 2x_m + x_{m-1} = h^2 \left[\sigma_0^* \ddot{x}_{m+1} + \sigma_1^* \nabla \ddot{x}_{m+1} + \sigma_2^* \nabla^2 \ddot{x}_{m+1} + \cdots + \sigma_k^* \nabla^k \ddot{x}_{m+1}\right]$$

(B) Equations for $\ddot{x} \rightarrow \dot{x}$

Predictor: (B1) $$\dot{x}_{m+1} - \dot{x}_m = h\left[\gamma_0 \ddot{x}_m + \gamma_1 \nabla \ddot{x}_m + \cdots + \gamma_k \nabla^k \ddot{x}_m\right]$$

Corrector: (B2) $$x_{m+1} - x_m = h\left[\gamma_0^* x_{m+1} + \gamma_1^* \nabla x_{m+1} + \cdots + \gamma_k^* \nabla^k x_{m+1}\right]$$

where the coefficients σ, σ^*, γ, γ^* are given by the following recurrence relationships:

*Henrici, 1962, pp 192-195 and 291-293.

letting $h_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$

$$\sigma_0 = 1, \quad \sigma_0^* = 1, \quad \gamma_0 = 1, \quad \gamma_0^* = 1$$

(1) $$\sigma_m = 1 - \frac{2}{3} h_2 \sigma_{m-1} - \frac{2}{4} h_3 \sigma_{m-2} - \cdots - \frac{2}{m+2} h_{m+1} \sigma_0$$

(2) $$\sigma_m^* = -\frac{2}{3} h_2 \sigma_{m-1}^* - \frac{2}{4} h_3 \sigma_{m-2}^* \cdots - \frac{2}{m+2} h_{m+1} \sigma_0^*$$

(3) $$\gamma_m = 1 - \frac{1}{2} \gamma_{m-1} - \frac{1}{3} \gamma_{m-2} \cdots - \frac{1}{m+1} \gamma_0$$

(4) $$\gamma_m^* = -\frac{1}{2} \gamma_{m-1}^* - \frac{1}{3} \gamma_{m-2}^* \cdots - \frac{1}{m+1} \gamma_0^*$$

$$m = 1, 2, 3, \cdots k$$

and $$\nabla \ddot{x}_m = \ddot{x}_m - \ddot{x}_{m-1}$$

$$\nabla^2 \ddot{x}_m = \nabla(\nabla \ddot{x}_m) = \nabla(\ddot{x}_m - \ddot{x}_{m-1}) = \ddot{x}_m - 2\ddot{x}_{m-1} + \ddot{x}_{m-1}$$

$$\nabla^k \ddot{x}_m = \nabla(\nabla^{k-1} \ddot{x}_m)$$

It has been established (Henrici, 1960, 1962) that an algebraic equivalent of equations (A) & (B), known as the "summed" form of the integration formulae, considerably reduces the propagation of round-off error. Formally, one can obtain this modification of equations (A) and (B) by applying the inverse difference operators ∇^{-1}, ∇^{-2} defined by $\nabla^{-1}\nabla = I$, $\nabla^{-2}\nabla^2 = I$, (I the identity), to both sides of these equations. In particular, by applying ∇^{-2} to both sides of equations (A) we obtain

Predictor: $$x_{m+1} = h^2 \left[\sigma_0 \nabla^{-2} \ddot{x}_m + \sigma_1 \nabla^{-1} \ddot{x}_m + \sigma_2 \ddot{x}_m + \right.$$

$$\left. \sigma_3 \nabla \ddot{x}_m + \cdots + \sigma_k \nabla^{k-2} \ddot{x}_m \right]$$

Corrector: $x_{m+1} = h^2 \left[\sigma_0^* \nabla^{-2}\ddot{x}_{m+1} + \sigma_1^* \nabla^{-1}\ddot{x}_{m+1} + \sigma_2^* \ddot{x}_{m-1} + \right.$

$\left. \sigma_3^* \nabla \ddot{x}_{m+1} + \cdots + \sigma_k^* \nabla^{k-2}\ddot{x}_{m+1}\right]$

If we define ${}^{I}S_m$, ${}^{II}S_m$ by

$$\begin{cases} \nabla^{-1}\ddot{x}_m = {}^{I}S_m , \\ \nabla\left({}^{I}S_m\right) = \ddot{x}_m \end{cases} \qquad \begin{cases} \nabla^{-2}\ddot{x}_m = \nabla^{-1}\left({}^{I}S_m\right) = {}^{II}S_m \\ \nabla^2\, {}^{II}S_m = \nabla\left({}^{I}S_m\right) = {}^{I}S_m - {}^{I}S_{m-1} = \ddot{x}_m \end{cases}$$

we have

Predictor: (A1)' $x_{m+1} = h^2 \left[\sigma_0\, {}^{II}S_m + \sigma_1\, {}^{I}S_m + \sigma_2 \ddot{x}_m + \cdots + \sigma_k \nabla^{k-2}\ddot{x}_m\right]$

and since

$$\nabla^2\, {}^{II}S_{m+1} = \nabla\left({}^{I}S_{m+1}\right) = {}^{I}S_{m+1} - {}^{I}S_m = \ddot{x}_{m+1}$$

we have ${}^{I}S_{m+1} = \ddot{x}_{m+1} + {}^{I}S_m$,

and since

$$\nabla\, {}^{II}S_{m+1} = {}^{II}S_{m+1} - {}^{II}S_m = {}^{I}S_{m+1}$$

we have

$$\begin{aligned} {}^{II}S_{m+1} &= {}^{II}S_m + {}^{I}S_{m+1} \\ &= {}^{II}S_m + {}^{I}S_m + x_{m+1} , \end{aligned}$$

so we have

Corrector: $$x_{m+1} = h^2 \left[\sigma_0^* \left({}^{II}S_m + {}^{I}S_m + \ddot{x}_{m+1}\right) + \sigma_1^* \left(\ddot{x}_{m+1} + {}^{I}S_m\right) + \sigma_2^* \ddot{x}_{m+1} + \sigma_3^* \nabla \ddot{x}_{m+1} + \cdots + \sigma_k^* \nabla^{k-2} \ddot{x}_{m+1}\right]$$

or

$$\text{(A2)}' \quad x_{m+1} = h^2 \left[\sigma_0^* \, {}^{II}S_m + \left(\sigma_0^* + \sigma_1^*\right) {}^{I}S_m + \left(\sigma_0^* + \sigma_1^* + \sigma_2^*\right) \ddot{x}_{m+1} + \sigma_3^* \nabla \ddot{x}_{m+1} + \cdots + \sigma_k^* \nabla^{k-2} \ddot{x}_{m+1}\right].$$

Likewise for equations (B), applying the operator ∇^{-1} to both sides, we obtain

Predictor: $$\dot{x}_{m+1} = h \left[\gamma_0 \nabla^{-1} \ddot{x}_{m+} + \gamma_1 \ddot{x}_m + \gamma_2 \nabla \ddot{x}_m + \cdots + \gamma_k \nabla^{k-1} \ddot{x}_m\right]$$

Corrector: $$\dot{x}_{m+1} = h \left[\gamma_0^* \nabla^{-1} \ddot{x}_{m+1} + \gamma_1^* \ddot{x}_{m+1} + \gamma_2^* \nabla \ddot{x}_{m+1} + \cdots + \gamma_k^* \nabla^{k-1} \ddot{x}_{m+1}\right]$$

and using the definitions of ${}^{I}S_m$ as above, we get

Predictor: $$\text{(B1)}' \quad \dot{x}_{m+1} = h \left[\gamma_0 \, {}^{I}S_m + \gamma_1 \ddot{x}_m + \gamma_2 \nabla \ddot{x}_m + \cdots + \gamma_k \nabla^{k-1} \ddot{x}_m\right]$$

and as before, since $\nabla \, {}^{I}S_{m+1} = {}^{I}S_{m+1} - {}^{I}S_m = \ddot{x}_{m+1}$

we have ${}^{I}S_{m+1} = \ddot{x}_{m+1} + {}^{I}S_m$

and hence

Corrector: $$\text{(B2)}' \quad \dot{x}_{m+1} = h \left[\gamma_0^* \left(\ddot{x}_{m+1} + {}^{I}S_m\right) + \gamma_1 \ddot{x}_{m+1} + \cdots + \gamma_k^* \nabla^{k-1} \ddot{x}_m\right]$$

$$\dot{x}_{m+1} = h \left[\gamma_0^* \, {}^{I}S_m + \left(\gamma_0^* + \gamma_1^*\right) \ddot{x}_{m+1} + \cdots + \gamma_k^* \nabla^{k-1} \ddot{x}_m\right]$$

These are the "summed" forms of the integration formulae. (For details concerning the computational usage of these formulae, see Reference 9.)

We remark again that equations (A1), (A2) and (A1)', (A2)' are algebraically equivalent, so that for any fixed k, any solution of (A1), (A2) is a solution of (A1)', (A2)', and vica versa; and in particular, the local truncation errors associated with these formulas are the same; (likewise for (B1), (B2) and (B1)', (B2)').

APPENDIX B

EQUATIONS OF MOTION

The equations of motion used to obtain the numerical results are given by

$$\ddot{\bar{x}} = -\frac{\mu x}{R^3} + \bar{F}_1 + \bar{F}_2$$

where $\bar{x} = (x, y, z)$, $R = ||\bar{x}|| - \left(x^2 + y^2 + z^2\right)^{1/2}$, μ is a constant, $\bar{F}_1$ the perturbation due to the non-sphericity of the earth, and $\bar{F}_2$ the perturbation due to drag. $\bar{F}_1 = \left(F_{1x}, F_{1y}, F_{1z}\right)$ is given by

$$F_{1x} = \frac{\mu x}{R^6}\left[JR^2(5s-1) + Hz(7s-3) + \frac{k}{6}\left(-63s^2 + 42s - 3\right)\right]$$

$$F_{1y} = \frac{\mu y}{R^6}\left[JR^2(5s-1) + Hz(7s-3) + \frac{k}{6}\left(-63s^2 + 42s - 3\right)\right]$$

$$F_{1z} = \frac{\mu z}{R^6}\left[JR^2(5s-1) + \frac{k}{6}\left(-63s^2 + 70s - 15\right)\right] + \frac{\mu H}{5R^5}\left(35s^2 - 30s + 3\right)$$

where $s = (z/R)^2$, J, H, K are the 2^{nd}, 3^{rd} and 4^{th} harmonics of the earth's potential field. $\bar{F}_2$ is of the form

$$\frac{C_D A}{2M}\rho||\bar{V}_r||\,\bar{V}_r$$

where V_r is the relative velocity of the satellite, ρ the atmospheric density at the satellite position; and A, M and C_D are the cross sectional area, mass and drag coefficient of the satellite respectively. The actual numerical values of these constants which were used to obtain the results can be found in Reference 9.